A description of quasi-duo Z-graded rings

André Leroy, Jerzy Matczuk, Puczylowski Edmund

▶ **To cite this version:**

André Leroy, Jerzy Matczuk, Puczylowski Edmund. A description of quasi-duo Z-graded rings. 2009. hal-00429128

HAL Id: hal-00429128
https://hal.science/hal-00429128

Preprint submitted on 30 Oct 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A description of quasi-duo $\mathbb{Z}$-graded rings

André Leroy[†], Jerzy Matczuk[*], Edmund R. Puczyłowski[*]

[†] Université d'Artois, Faculté Jean Perrin
Rue Jean Souvraz 62 307 Lens, France
e.mail: leroy@poincare.univ-artois.fr

[*] Institute of Mathematics, Warsaw University,
Banacha 2, 02-097 Warsaw, Poland
e.mail: jmatczuk@mimuw.edu.pl, edmundp@mimuw.edu.pl

Abstract

A description of right (left) quasi-duo $\mathbb{Z}$-graded rings is given. It shows, in particular, that a strongly $\mathbb{Z}$-graded ring is left quasi-duo if and only if it is right quasi-duo. This gives a partial answer to a problem posed by Dugas and Lam in [1].

A ring R with an identity is called [1] *right* (*left*) *quasi-duo* if every maximal right (left) ideal of R is two-sided. Quasi-duo rings were studied in many papers (Cf. [1], [5] and papers quoted there). The main open problem in the area asks whether the classes of left and right quasi-duo rings coincide (it is important, as it concerns the problem to what extend the notion of primitivity is left-right symmetric, Cf. [1]). This problem was also an initial motivation for our studies. Namely the results obtained in [2] on quasi-duo skew polynomial rings show that it would be interesting to examine whether it could be possible to distinct these classes within $\mathbb{Z}$-graded rings or, more generally, to describe $\mathbb{Z}$-graded right (left) quasi-duo rings. The methods of [2] are rather specific for skew-polynomial rings and one cannot apply them to $\mathbb{Z}$-graded rings. In this paper we find another approach to that problem and describe $\mathbb{Z}$-graded right (left) quasi-duo rings. This description shows, in particular, that a strongly $\mathbb{Z}$-graded ring is right quasi-duo if and only if it is left quasi-duo. Thus, for strongly $\mathbb{Z}$-graded rings, the above mentioned Dugas-Lam problem has a positive solution. As an application we also get back in another way the characterization of right (left) skew polynomial and Laurent polynomial rings obtained in [2].

The results on the Jacobson radical, the pseudoradical and maximal ideals of $\mathbb{Z}$-graded rings (see Proposition 3, Theorem 2) can be of independent interest.

All rings in this paper are associative with identity. To denote that I is an ideal (left ideal, right ideal) of a ring R we will write $I \lhd R$ ($I <_l R$, $I <_r R$). The Jacobson radical of a ring R will be denoted by $J(R)$.

It is clear that R is right (left) quasi-duo if and only if $R/J(R)$ is right (left) quasi-duo and that Jacobson semisimple right (left) quasi-duo rings are subdirect sums of division rings, so they are reduced rings. The class of right (left) quasi-duo rings is closed under homomorphic images and finite subdirect sums (Cf.[1]).

In what follows $\mathbb{Z}$ denotes the additive group of integers and R denotes a $\mathbb{Z}$-graded ring. Recall that $R = \bigoplus_{n\in\mathbb{Z}} R_n$, the direct sum of additive subgroups R_n, with $R_nR_m \subseteq R_{n+m}$ for all $n, m \in \mathbb{Z}$. If $R_nR_m = R_{n+m}$, then R is called *strongly graded*.

[*]The research was supported by Polish MNiSW grant No. N N201 268435

Elements of $\bigcup_{n\in\mathbb{Z}} R_n$ are called *homogeneous.* Every $r \in R$ can be written as a finite sum $r = \sum_{m\leq i\leq n} r_i$, where $r_i \in R_i$ is called the *homogeneous component* of r of degree i. If r_m and r_n are nonzero, then the length $l(r)$ of r is defined as $n - m + 1$. Clearly a nonzero element of R is homogeneous if and only if its length is equal to 1.

An ideal I of R is called *homogeneous* if $I = \bigoplus_{n\in\mathbb{Z}}(I \cap R_n)$. The largest homogeneous ideal contained in a given ideal I of R will be denoted by $(I)_h$.

The following well known result of G. Bergman (Cf. [4]) plays a substantial role in the paper.

Theorem 1. *For every $\mathbb{Z}$-graded ring R*

(i) $J(R)$ is a homogeneous ideal;

(ii) If $r \in \bigcup_{0\neq n\in\mathbb{Z}} R_n$, then $1 + r$ is invertible if and only if r is nilpotent.

A homogeneous ideal P of R is called *graded prime* if $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$ for arbitrary homogeneous ideals I and J of R. It is well known and not hard to check that if P is a prime ideal of R, then $(P)_h$ is a graded prime ideal of R. It is also well known that a homogeneous ideal of a $\mathbb{Z}$-graded ring is prime if and only if it is graded prime.

The intersection of all nonzero graded prime ideals of R will be called the *graded pseudoradical* of R. The empty intersection, by definition, is equal to R.

The following result generalizes Lemma 3.2 from [3].

Theorem 2. *Suppose that a $\mathbb{Z}$-graded ring R contains a maximal ideal M such that $(M)_h = 0$. Then the graded pseudoradical of R is nonzero.*

Proof. Let $a = \sum_{m\leq i\leq n} a_i$ be a nonzero element of M of minimal length, where $a_m \neq 0 \neq a_n$. Since $(M)_h = 0$, $l(a) \geq 2$.

Let C (resp. D) denote the sets of all n-th (resp. m-th) components of nonzero elements from $M \cap (\bigoplus_{m\leq i\leq n} R_i)$. Notice that C and D are non empty homogeneous sets depending only on M.

If R has no nonzero graded prime ideals, then the graded pseudoradical of R is equal to R, so the thesis holds.

Suppose now that we can pick a nonzero graded prime ideal Q of R. Then $M + Q = R$, so $1 = b + q$, where $b = \sum_{s\leq l\leq t} b_l \in M$ and $1 - b_0 \in Q$ and $b_i \in Q$, for $s \leq i \leq t$, $i \neq 0$. This implies that precisely one homogeneous component of $b = \sum_{s\leq l\leq t} b_l \in M$ is not in Q. Suppose that b is an element in M with the smallest possible length amongst the elements of M having precisely one homogeneous component not in Q. Let us write $b = \sum_{s\leq l\leq t} b_l \in M$ with $b_k \notin Q$.

If $k \neq t$ we claim that $C \subseteq Q$. If not then there exists $r = \sum_{m\leq i\leq n} r_i \in M$ such that $r_n \notin Q$. Since Q is a prime graded ideal, there is $c \in R_w$, for some $w \in \mathbb{Z}$, such that $b_k c r_n \notin Q$. Notice that $n - m + 1 = l(r) \leq l(b) = t - s + 1$ and the element $u = bcr_n - b_t cr \in M$ is such that precisely one homogeneous component of u (namely u_{k+w+n}) is not in Q. Moreover, since $(bcr_n)_l = (b_t cr)_l = 0$ if $l < s + w + n$ and $u_{t+w+n} = 0$, we get $l(u) < l(b)$, which is impossible, by the choice of b. This proves the claim.

If $k = t$ we can prove in a similar way that $D \subseteq Q$.

We conclude that $CRD \subseteq Q$ for any nonzero graded prime ideal Q of R. Since M is prime and $(M)_h = 0$, the ring R is a graded prime ring and hence $CRD \neq 0$. This yields the desired result. □

In what follows we denote by $\mathcal{A}$ the set of all maximal right ideals M of R such that $R_n \not\subseteq M$, for some $0 \neq n \in \mathbb{Z}$ and by $\mathcal{B}$ the set of remaining maximal right ideals of R. Set $A(R) = \bigcap_{M\in\mathcal{A}} M$ and $B(R) = \bigcap_{M\in\mathcal{B}} M$.

It is easy to describe $B(R)$. Note that $U = \sum_{0\neq n\in\mathbb{Z}} R_{-n}R_n \lhd R_0$. It is clear that if $M \in \mathcal{B}$, then $M = M_0 + \bigoplus_{0\neq n\in\mathbb{Z}} R_n$ for a maximal right ideal M_0 of R_0 containing U. Consequently $B(R) = J + \sum_{0\neq n\in\mathbb{Z}} R_n$, where J is the ideal of R_0 containing U such that $J(R_0/U) = J/U$. In particular, $B(R)$ is a two-sided ideal of R.

If R is strongly graded, then for every $0 \neq n \in \mathbb{Z}$, $R_0 = R_nR_{-n}$. This shows that in this case $\mathcal{B} = \emptyset$, so $B(R) = R$ and $A(R) = J(R)$.

Now we will describe $A(R)$. Let $A_l = \{r \in R \mid R_n r \subseteq J(R)$, for every $0 \neq n \in \mathbb{Z}\}$ and $A_r = \{r \in R \mid rR_n \subseteq J(R)$, for every $0 \neq n \in \mathbb{Z}\}$.

Proposition 3. *Let R be a $\mathbb{Z}$-graded ring. Then:*

(i) $A(R) = A_l = A_r$

(ii) $A(R) \cap (\bigoplus_{0\neq n\in\mathbb{Z}} R_n) = J(R) \cap (\bigoplus_{0\neq n\in\mathbb{Z}} R_n)$.

Proof. (i). It is clear that $A_l \lhd R$. Hence $A_lR_n <_l R$, for every $0 \neq n \in \mathbb{Z}$. Since $(A_lR_n)^2 \subseteq J(R)$ and $R/J(R)$ is semiprime, $A_lR_n \subseteq J(R)$. This proves that $A_l \subseteq A_r$. Dual arguments give the opposite inclusion and show that $A_l = A_r$.

Take any $M \in \mathcal{A}$. Then $R_n \not\subseteq M$, for some $0 \neq n \in \mathbb{Z}$. Obviously $(A_r + M)R_n \subseteq M$. Thus $A_r + M \neq R$ and maximality of M implies that $A_r \subseteq M$. Consequently $A_r \subseteq A(R)$. Clearly $A(R) \cap B(R) = J(R)$, $B(R) \lhd R$ and $A(R) <_r R$, so $A(R)B(R) \subseteq J(R)$. Hence, since $\bigoplus_{0\neq n\in\mathbb{Z}} R_n \subseteq B(R)$, we get that $A(R) \subseteq A_r$.

(ii). By (i), $A(R)R_n + R_mA(R) \subseteq J(R)$, for arbitrary $n, m \in \mathbb{Z} \setminus \{0\}$. This implies that if I is the ideal of R generated by $A(R) \cap (\bigoplus_{0\neq n\in\mathbb{Z}} R_n)$, then $I^2 \subseteq J(R)$. Consequently $A(R) \cap (\bigoplus_{0\neq n\in\mathbb{Z}} R_n) \subseteq I \subseteq J(R)$. Now it is easy to complete the proof of (ii). □

Theorem 4. *If a $\mathbb{Z}$-graded ring R is right (left) quasi-duo, then R/M is a field, for every $M \in \mathcal{A}$.*

Proof. We will prove the result when R is right quasi-duo. If R is left quasi-duo, symmetric arguments can be applied. Let $M \in \mathcal{A}$. Passing to the factor ring $R/(M)_h$, we can assume without loss of generality that $(M)_h = 0$. Since R is right quasi-duo, R/M is a division ring. Making use of those two facts, one can easily check that R is a domain. Moreover, by Theorem 2, the graded pseudoradical P of R is nonzero.

Let $0 \neq n \in \mathbb{Z}$ and $a \in P_n = P \cap R_n$. Clearly a is not nilpotent, as R is a domain. Thus, by Theorem 1, $1+a$ is not invertible. Hence there exists a maximal right ideal T of R containing $1+a$. Since R is quasi-duo, $T \lhd R$. Now $(T)_h$ is a prime homogeneous ideal of R, so if $(T)_h \neq 0$, then $P \subseteq T$. This is impossible as otherwise $1 = (1+a) - a \in T$. Therefore $(T)_h = 0$. Now for every homogeneous element b of R, $ab - ba = (1+a)b - b(1+a) \in (T)_h = 0$. This shows that a belongs to the center $Z(R)$ of R and implies that $P_n \subseteq Z(R)$, for all nonzero $n \in \mathbb{Z}$. Since $M \in \mathcal{A}$, by definition, there exists $0 \neq m \in \mathbb{Z}$ such that $R_m \not\subseteq M$. In particular $R_m \neq 0$. Therefore, since P is a nonzero homogeneous ideal and R is a domain, we can pick a nonzero integer n such that $P_n \neq 0$. Then $P_0P_n \subseteq P_n \subseteq Z(R)$ and, as R is a domain, $P_0 \subseteq Z(R)$ follows. The above implies that $P \subseteq Z(R)$ and shows that the division ring $R/M = (M+P)/M$ is commutative, i.e. it is a field. □

Theorem 5. *A $\mathbb{Z}$-graded ring R is right (left) quasi-duo if and only if R_0 is right (left) quasi-duo and $R/A(R)$ is a commutative ring.*

Proof. Suppose that R is right quasi-duo. Let M be a maximal right ideal of R_0. Clearly MR is a proper right ideal of R. Consequently MR is contained in a maximal right ideal T of R. Since R is right quasi-duo, $T \lhd R$. It is clear that $M = T \cap R_0$, so $M \lhd R_0$. Thus R_0 is a right quasi-duo ring.

When $\mathcal{A} \neq \emptyset$, Theorem 4 implies that $R/A(R)$ is a subdirect sum of fields, so it is a commutative ring. If $\mathcal{A} = \emptyset$, then $A(R) = R$ and the ring $R/A(R)$ is also commutative.

Suppose now that R_0 is right quasi-duo and $R/A(R)$ is commutative. Let I be the ideal of R generated by $\bigcup_{0\neq n\in\mathbb{Z}} R_n$. Then, by Proposition 3(i), $IA(R) \subseteq J(R)$. Hence $(I \cap A(R))^2 \subseteq J(R)$ and semiprimeness of $J(R)$ implies that $I \cap A(R) \subseteq J(R)$. This shows that $R/J(R)$ is a homomorphic image of a subdirect sum of rings R/I and $R/A(R)$. Clearly R/I is a homomorphic image of R_0. Consequently both R/I and $R/A(R)$ are right quasi-duo, so, further, $R/J(R)$ and R are right quasi-duo.

When R is left quasi-duo, symmetric arguments apply. □

Theorem 5 immediately gives the following

Corollary 6. *Suppose a $\mathbb{Z}$-graded ring R is right quasi-duo. Then:*
1. R_0 is right quasi-duo;
2. R is left quasi-duo iff R_0 is left quasi-duo.

We know, by the remark made just before Proposition 3, that $A(R) = J(R)$, provided R is strongly $\mathbb{Z}$-graded. Thus, by Theorem 5, we get:

Corollary 7. *Suppose that R is strongly $\mathbb{Z}$-graded. Then R is right quasi-duo iff R is left quasi-duo iff $R/J(R)$ is commutative.*

Now, as an application of Theorem 5, we will get characterizations of right (left) quasi-duo skew polynomial rings and skew Laurent polynomial rings obtained in [2].

Let σ be an endomorphism of a ring S and $S[x;\sigma]$ be the associated skew polynomial ring with coefficients from S written on the left. Denote by $N(S)$ the set $\{s \in S \mid s\sigma(s)\cdots\sigma^n(s) = 0$, for some positive integer $n\}$. Clearly $N(S) = \{s \in S \subseteq S[x;\sigma] \mid (sx)^n = 0$, for some positive integer $n\}$. Let $N(S)[x;\sigma]$ be the set of all polynomials from $S[x;\sigma]$ which have all their coefficients in $N(S)$. Notice also that $\sigma(N(S)) \subseteq N(S)$. Thus, if $N(S) \lhd S$ then $N(S)[x;\sigma] \lhd S[x;\sigma]$, σ induces an endomorphism, also denoted by σ, on $S/N(S)$ and $(S/N(S))[x;\sigma] \simeq S[x;\sigma]/N(S)[x;\sigma]$.

Lemma 8. *Suppose that the skew polynomial ring $S[x;\sigma]$ is right (left) quasi-duo. Then $J(S[x;\sigma]) \subseteq N(S)[x;\sigma] \subseteq A(S[x;\sigma])$.*

Proof. Since $S[x;\sigma]$ is right (left) quasi-duo, the ring $S[x;\sigma]/J(S[x;\sigma])$ is reduced, so every nilpotent element of $S[x;\sigma]$ belongs to $J(S[x;\sigma])$. Thus, in particular, $xN(S) \subseteq J(S[x;\sigma])$ and consequently $Sx^nN(S) \subseteq J(S[x;\sigma])$, for all $n > 0$. The ring $S[x;\sigma]$ is $\mathbb{Z}$-graded in the canonical way and the last inclusion together with Proposition 3(i) yield $N(S) \subseteq A(S[x;\sigma])$. This shows that $N(S)[x;\sigma] \subseteq A(S[x;\sigma])$.

Let $ax^n \in J(S[x;\sigma])$, for some $n > 0$. Then, by Theorem 1, ax^n and x^na are also nilpotent elements of $S[x;\sigma]$ and so $x^na \in J(S[x;\sigma])$. Hence $Sx^mx^{n-1}a \subseteq J(S[x;\sigma])$, for all $m > 0$ and Proposition 3(i) shows that $x^{n-1}a \in J(S[x;\sigma])$. Repeating this procedure we obtain $xa \in J(S[x;\sigma])$ and Theorem 1 implies that $a \in N(S)$. Since $J(S[x;\sigma])$ is a homogenous ideal, we obtain $J(S[x;\sigma]) \subseteq N(S)[x;\sigma]$. □

Corollary 9. ([2]) *$S[x;\sigma]$ is right (left) quasi-duo if and only if S is right (left) quasi-duo, $N(S) \lhd S$, $J(S[x;\sigma]) = J(S) \cap N(S) + N(S)[x;\sigma]x$ and $(S/N(S))[x;\sigma]$ is a commutative ring.*

Proof. Suppose that the ring $S[x;\sigma]$ is right (left) quasi-duo. Then, by Proposition 3(i), $A(S[x;\sigma]x) \subseteq J(S[x;\sigma])$. Thus, by Lemma 8, we get $A(S[x;\sigma]) = N(S)[x;\sigma]$. This implies that $N(S)$ is an ideal of S. Now, by Theorem 5, the ring $(S/N(S))[x;\sigma] \simeq S[x;\sigma]/N(S)[x;\sigma]$ is commutative.

Since $B(S[x;\sigma]) = J(S) + S[x;\sigma]x$ and $J(T) = A(T) \cap B(T)$, we also obtain $J(S[x;\sigma]) = J(S) \cap N(S) + N(S)[x;\sigma]x$.

Conversely, by making use of Proposition 3(i), it is evident that when $J(S[x;\sigma]) = J(S) \cap N(S) + N(S)[x;\sigma]$, then $A(S[x;\sigma]) = N(S)[x;\sigma]$. Now if the ring $(S/N(S))[x;\sigma]$ is commutative and S is right (left) quasi-duo, then $S[x;\sigma]$ is right (left) quasi-duo, by Theorem 5. □

Corollary 10. ([2]) *Let σ be an automorphism of a ring S. Then the skew Laurent polynomial ring $S[x, x^{-1};\sigma]$ is right (left) quasi-duo if and only if $N(S) \lhd S$, $J(S[x, x^{-1};\sigma]) = N(S)[x, x^{-1};\sigma]$ and $(S/N(S))[x, x^{-1};\sigma]$ is a commutative ring.*

Proof. Since $S[x, x^{-1};\sigma]$ is a strongly graded, $A(S[x, x^{-1};\sigma]) = J(S[x, x^{-1};\sigma])$.

Suppose now that $S[x, x^{-1};\sigma]$ is right (left) quasi-duo. Then, as $N(S)x$ consists of nilpotent elements, $N(S)[x, x^{-1};\sigma] \subseteq J(S[x, x^{-1};\sigma])$. The opposite inclusion follows immediately from Theorem 1. Obviously $N(S) \lhd S$ and by Theorem 5, $(S/N(S))[x, x^{-1};\sigma]$ is a commutative ring. This proves the only if" part. The "if" part is a direct consequence of Theorem 5. □

References

[1] T.Y. Lam, A.S. Dugas, Quasi-duo rings and stable range descent, J. Pure and Appl. Alg. 195 (2005) 243–259.

[2] A. Leroy, J. Matczuk and E.R. Puczyłowski, Quasi-duo skew polynomial rings, J. Pure and Appl. Alg. 212, 2008, 1951–1959.

[3] J. Matczuk, Maximal ideals of skew polynomial rings of automorphism type, Comm. Algebra 24(3), 1996, 907–917.

[4] L.H. Rowen, *Ring Theory, vol. I,* Pure and Applied Mathematics vol. 127, Academic Press, 1988.

[5] H. P. Yu, On quasi-duo rings, Glasgow Math. J. 37, 1995, 21–31.